Properness for iterations with uncountable supports

based on joint works of
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presented by AR

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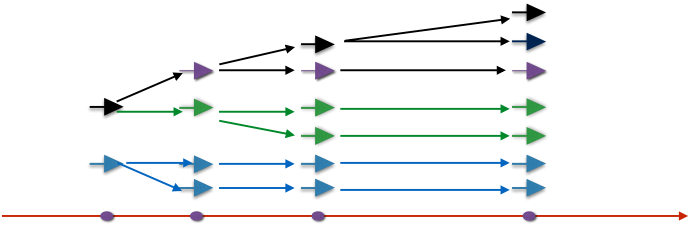
Trees of conditions

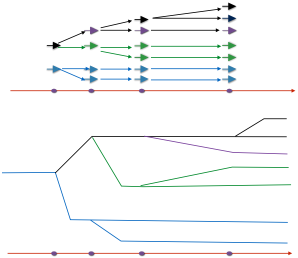
Let γ be an ordinal, $\emptyset \neq w \subseteq \gamma$. *A standard* $(w,1)^\gamma$*–tree* is a pair $\mathcal{T} = (T, \mathrm{rk})$ such that

- $\mathrm{rk} : T \longrightarrow w \cup \{\gamma\}$,
- if $t \in T$ and $\mathrm{rk}(t) = \varepsilon$, then t is a sequence $\langle (t)_\zeta : \zeta \in w \cap \varepsilon \rangle$,
- $(T, \lhd)$ is a tree with root $\langle\rangle$ and such that every chain in T has a $\lhd$–upper bound it T,
- if $t \in T$, then there is $t' \in T$ such that $t \unlhd t'$ and $\mathrm{rk}(t') = \gamma$.



Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \dot{\mathbb{Q}}_i : i < \gamma \rangle$ be an iteration.

◇ *A standard tree of conditions in* $\bar{\mathbb{Q}}$ is a system $\bar{p} = \langle p_t : t \in T \rangle$ such that



- (T, rk) is a standard $(w, 1)^{\gamma}$–tree for some $w \subseteq \gamma$,
- $p_t \in \mathbb{P}_{\mathrm{rk}(t)}$ for $t \in T$, and
- if $s, t \in T$, $s \lhd t$, then $p_s = p_t \restriction \mathrm{rk}(s)$.

◇ Let $\bar{p}^0, \bar{p}^1$ be standard trees of conditions in $\bar{\mathbb{Q}}$, $\bar{p}^i = \langle p^i_t : t \in T \rangle$. We write $\bar{p}^0 \leq \bar{p}^1$ whenever for each $t \in T$ we have $p^0_t \leq p^1_t$.

Theorem 1

Assume that $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \mathop{\mathbb{Q}}\limits_{\sim}{}_i : i < \gamma \rangle$ *is a* λ*–support iteration such that for all* $i < \gamma$ *we have*

$$\Vdash_{\mathbb{P}_i} \text{“} \mathop{\mathbb{Q}}\limits_{\sim}{}_i \text{ is strategically } ({<}\lambda)\text{–complete ”.}$$

Suppose that $\bar{p} = \langle p_t : t \in T \rangle$ *is a standard tree of conditions in* $\bar{\mathbb{Q}}$*,* $|T| < \lambda$*, and* $\mathcal{I} \subseteq \mathbb{P}_\gamma$ *is open dense. Then there is a standard tree of conditions* $\bar{q} = \langle q_t : t \in T \rangle$ *such that* $\bar{p} \leq \bar{q}$ *and* $(\forall t \in T)(\mathrm{rk}(t) = \gamma \ \Rightarrow \ q_t \in \mathcal{I})$*.*

A very simple yet informative property in CS

We will the main ideas of our bounding properties by looking at their ω–relative. I do not know if the *strong bounding* introduced here is of any use, but it explains nicely what is going on in the λ–case.

Let $\mathbb{P}$ be a forcing notion and $p \in \mathbb{P}$.

We define a game $\Game^{\rm sb}(p, \mathbb{P})$ between two players, Generic and Antigeneric, as follows. A play of $\Game^{\rm sb}(p, \mathbb{P})$ lasts ω steps and during the play a sequence

$$\bar{x} = \left\langle m_k, \langle p^k_\ell, q^k_\ell : \ell < m_k \rangle : k < \omega \right\rangle$$

is constructed.

Suppose that the players have arrived at a stage $k < \omega$ of the game. Now,

$(\aleph)_k$ first Generic chooses a positiv integer m_k and a sequence $\langle p^k_\ell : \ell < m_k \rangle$ of conditions from $\mathbb{P}$.

$(\beth)_k$ Then Antigeneric answers by picking a system $\langle q^k_\ell : \ell < m_k \rangle$ of conditions from $\mathbb{P}$ such that $p^k_\ell \leq q^k_\ell$ (for all $\ell < m_k$).

At the end, Generic wins the play $\bar{x}$ iff

$(\circledast)$ there is a condition p^* stronger than p such that

$$p^* \Vdash_{\mathbb{P}} (\forall k < \omega)(\exists \ell < m_k)(q^k_\ell \in \underset{\sim}{G}_{\mathbb{P}}).$$

We say that $\mathbb{P}$ is *strongly bounding* if for any $p \in \mathbb{P}$ Generic has a winning strategy in $\Game^{\rm sb}(p, \mathbb{P})$.

Observation 2

(1) The Sacks forcing notion is strongly bounding while the random real forcing is not.
(2) Every strongly bounding forcing is proper and ${}^{\omega}\omega$–bounding.

Theorem 3

Assume that $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma \rangle$ is a countable support iteration such that for every $\xi < \gamma$,

$$\Vdash_{\mathbb{P}_\xi} \text{“} \underset{\sim}{\mathbb{Q}}_\xi \text{ is strongly bounding ”.}$$

Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ is proper ${}^{\omega}\omega$–bounding and more (see the proof and later).

Proof

We will present the key construction of the proof in the form of a play of a game (without exactly describing the rules of the game). This way we will set the ground for explaining what is the meaning of the "and more". The players are called **G** and **A**. Let $\langle D_k : k < \omega\rangle$ be a list of open dense subsets of $\mathbb{P}_\gamma$ (e.g., the list of all such sets from a model N) and let $p \in \mathbb{P}_\gamma$.

In the game/construction for $k < \omega$,

first **G** picks: $\mathcal{T}_k, \bar{p}^k$, $\langle \underset{\sim}{m}_{k,\xi}, \underset{\sim}{\bar{p}}_{k,\xi}, \underset{\sim}{\bar{q}}_{k,\xi} : \xi \in w_k\rangle$

then **A** answers with $\bar{q}^k$

and next **G** decides r_{k+1}, w_{k+1} and $\underset{\sim}{\mathbf{st}}_\xi$ for $\xi \in w_{k+1} \setminus w_k$

These objects are chosen so that for each $k < \omega$:

Choice of **G***:*

$(*)_1$ $r_k \in \mathbb{P}_\gamma$, we stipulate $r_{-1} = p$ and then $r_{-1} \leq r_k \leq r_{k+1}$, $r_0(0) = p(0)$ and $r_k(\xi) = r_{k+1}(\xi)$ for $\xi \in w_k$.

$(*)_2$ $w_k \subseteq \gamma$, $|w_k| = |k+1|$, $\bigcup\limits_{k<\omega} \mathrm{Dom}(r_k) = \bigcup\limits_{k<\omega} w_k$, $w_0 = \{0\}$, $w_k \subseteq w_{k+1}$

$(*)_3$ $\mathbf{st}_0$ is a winning strategy of Generic in $\Game^{\rm sbg}(r_0(0), \mathbb{Q}_0)$ and if $\xi \in w_{k+1} \setminus w_k$, then $\underset{\sim}{\mathbf{st}}_\xi$ is a $\mathbb{P}_\xi$–name for a winning strategy of Generic in $\Game^{\rm sbg}(r_{k+1}(\xi), \underset{\sim}{\mathbb{Q}}_\xi)$. We assume that these strategies instruct Generic to play conditions compatible with $r_{k+1}(\xi)$.

$(*)_4$ $\mathcal{T}_k = (T_k, \mathrm{rk}_k)$ is a finite standard $(w_k, 1)^\gamma$–tree, and $\bar{p}^k = \langle p^k_t : t \in T_k \rangle$ is a standard trees of conditions in $\bar{\mathbb{Q}}$.

$(*)_5$ If $\xi \in w_k$, then $\underset{\sim}{m}_{k,\xi}$ is a $\mathbb{P}_\xi$–name for a positive ordinal, $\underset{\sim}{\bar{p}}_{k,\xi}, \underset{\sim}{\bar{q}}_{k,\xi}$ are $\mathbb{P}_\xi$–names for $\underset{\sim}{m}_{k,\xi}$–sequences of conditions in $\underset{\sim}{\mathbb{Q}}_\xi$.

$(*)_6$ If $\xi \in w_{\ell+1} \setminus w_\ell$ and $\ell < \omega$, or $\xi = 0$ and $\ell = -1$, then

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \langle \underset{\sim}{m}_{n,\xi}, \underset{\sim}{\bar{p}}_{n,\xi}, \underset{\sim}{\bar{q}}_{n,\xi} : \ell < n < \omega \rangle \text{ is a } \textit{tail of a play} \text{ of } \Game^{\rm sbg}(r_\ell(\xi), \underset{\sim}{\mathbb{Q}}_\xi) \text{ in which Generic uses } \underset{\sim}{\mathbf{st}}_\xi \text{ ”.}$$

By “tail of a play” I mean that it can be completed to a full play in which Generic uses her strategy $\underset{\sim}{\mathbf{st}}_\xi$ and an the initial stages Antigeneric just repeats her entries.

$(*)_7$ For $t \in T_k$ we have $r_{k-1} \restriction \mathrm{rk}_k(t) \leq p^k_t$.

$(*)_8$ If $t \in T_k$, $\mathrm{rk}_k(t) = \xi < \gamma$, then the condition p^k_t decides the value of $\underset{\sim}{m}_{k,\xi}$, say $p^k_t \Vdash$“$\underset{\sim}{m}_{k,\xi} = m^t_{k,\xi}$”, and $\{(s)_\xi : t \lhd s \in T_k\} = m^t_{k,\xi}$ and $p^k_t \Vdash_{\mathbb{P}_\xi} \underset{\sim}{\bar{p}}_{k,\xi}(i) \leq p^k_{t^\frown\langle i\rangle}(\xi)$

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$$p^k_{t_0 \restriction \xi} \Vdash_{\mathbb{P}_\xi} \text{“ the conditions } p^k_{t_0}(\xi), p^k_{t_1}(\xi) \text{ are incompatible ”.}$$

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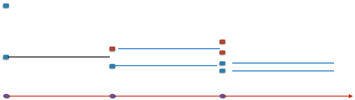

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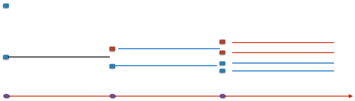

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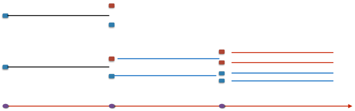

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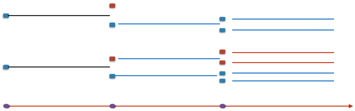

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Choice of **A**:

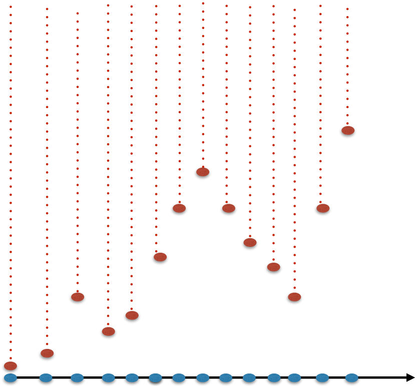
$(*)_{10}$ $\bar{q}^k = \langle q_t^k : t \in T_k \rangle$ is a standard tree of conditions in $\bar{\mathbb{Q}}$, $\bar{p}^k \le \bar{q}^k$, and $q_t^k \in D_k$ for all $t \in T_k$ of rank γ.

$(*)_{11}$ $q_t^k \Vdash_{\mathbb{P}_\xi}$ " $\underset{\sim}{\bar{q}}_{k,\xi}(i) = q^k_{t^\frown\langle i \rangle}(\xi)$ for $i < m^t_{k,\xi}$ " for all $t \in T_k$.

Choice of **G**:

$(*)_{12}$ $\mathrm{Dom}(r_k) = \bigcup_{t \in T_k} \mathrm{Dom}(q_t^k) \cup \mathrm{Dom}(p)$ and if $t \in T_k$, $\xi \in \mathrm{Dom}(r_k) \cap \mathrm{rk}_k(t) \setminus w_k$, and $q_t^k \restriction \xi \le q \in \mathbb{P}_\xi$, $r_k \restriction \xi \le q$, then

$q \Vdash_{\mathbb{P}_\xi}$ " if the set $\{r_\ell(\xi) : \ell < k\} \cup \{q_t^k(\xi), p(\xi)\}$
has an upper bound in $\underset{\sim}{\mathbb{Q}}_\xi$,
then $r_k(\xi)$ is such an upper bound ".



After the game/construction is over, define a condition $r \in \mathbb{P}_\gamma$ as follows. Let $\mathrm{Dom}(r) = \bigcup_{k<\omega} \mathrm{Dom}(r_k)$ and for $\xi \in \mathrm{Dom}(r)$ let $r(\xi)$ be a $\mathbb{P}_\xi$–name for a condition in $\mathbb{Q}_\xi$ such that if $\xi \in w_{\ell+1} \setminus w_\ell$, $\ell < \omega$ (or $\xi = 0$ and $\ell = -1$), then

$$\Vdash_{\mathbb{P}_\xi} \text{“ } r(\xi) \geq r_\ell(\xi) \text{ and } r(\xi) \Vdash_{\underset{\sim}{\mathbb{Q}}_\xi} (\forall k \in (\ell, \omega))(\exists i < \underset{\sim}{m}_{k,\xi})(\bar{q}_{k,\xi}(i) \in \underset{\sim}{G}_{\underset{\sim}{\mathbb{Q}}_\xi}) \text{ ”.}$$

Then $r \geq p$ and for each $k < \omega$ the family $\{q^k_t : t \in T_k \ \&\ \mathrm{rk}_k(t) = \gamma\}$ is pre-dense above r.

What have we actually proved?

- We cannot say that $\mathbb{P}_\gamma$ is strongly bounding as the game changes. We play with *trees of conditions*! (I.e., Antigeneric has to answer with such).
- We may argue for a game in which "maximal braches of the tree T_k" and corresponding conditions p_t^k are played successively forcing Antigeneric to build something close to a tree of conditions.
- If we want a *preservation theorem* then we need to modify the game mentioned above even further allowing several "runs" of the successive choices above.

The real stuff: the As

In this part we assume the following:

(a) λ is a strongly inaccessible cardinal,

(b) $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$, each μ_α is a regular cardinal satisfying (for $\alpha < \lambda$)

$$\aleph_0 \leq \mu_\alpha \leq \lambda \qquad \text{and} \qquad (\forall f \in {}^{\alpha}\mu_\alpha)\big(\big|\prod_{\xi<\alpha} f(\xi)\big| < \mu_\alpha\big),$$

(c) $\varphi : \lambda \longrightarrow \lambda$ is a strictly increasing function such that $\aleph_0 + \alpha < \varphi(\alpha)$ is regular,

(d) $\bar{F} = \langle F_t : t \in \bigcup_{\alpha<\lambda} \prod_{\beta<\alpha} \varphi(\beta) \rangle$ where F_t is a $<\varphi(\alpha)$–complete filter on $\varphi(\alpha)$ whenever $t \in \prod_{\beta<\alpha} \varphi(\beta)$, $\alpha < \lambda$.

(e) $\bar{E} = \langle E_t : t \in {}^{<\lambda}\lambda \rangle$ is a system of $(<\lambda)$–complete filters on λ.

(f) E is a normal filter on λ.

Game A

Let $p \in \mathbb{Q}$. We define a game $\Game^{\rm rcA}_{\bar{\mu}}(p, \mathbb{Q})$ between two players, Generic and Antigeneric, as follows. A play of $\Game^{\rm rcA}_{\bar{\mu}}(p, \mathbb{Q})$ lasts λ steps and during a play a sequence

$$\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha\rangle : \alpha < \lambda\Big\rangle$$

is constructed. At stage $\alpha < \lambda$ of the game:

$(\aleph)_\alpha$ first Generic chooses a non-empty set I_α of cardinality $< \mu_\alpha$ and a system $\langle p^\alpha_t : t \in I_\alpha\rangle$ of conditions from $\mathbb{Q}$,

$(\beth)_\alpha$ then Antigeneric answers by picking a system $\langle q^\alpha_t : t \in I_\alpha\rangle$ of conditions from $\mathbb{Q}$ such that $(\forall t \in I_\alpha)(p^\alpha_t \leq q^\alpha_t)$.

At the end, Generic wins the play

$$\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha\rangle : \alpha < \lambda\Big\rangle$$

of $\Game^{\rm rcA}_{\bar{\mu}}(p, \mathbb{Q})$ if and only if

$(\circledast)^{\rm rc}_{\rm A}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \big(\forall \alpha < \lambda\big)\big(\exists t \in I_\alpha\big)\big(q^\alpha_t \in \mathop{G}\limits_{\sim}{}_{\mathbb{Q}}\big).$$

Game a

Let $p \in \mathbb{Q}$. A game $\Game^{\rm rc\mathbf{a}}_{\bar{\mu}}(p, \mathbb{Q})$ between Generic and Antigeneric is defined as follows. A play of $\Game^{\rm rc\mathbf{a}}_{\bar{\mu}}(p, \mathbb{Q})$ lasts λ steps and during a play a sequence

$$\Big\langle \zeta_\alpha, \langle p^\alpha_\xi, q^\alpha_\xi : \xi < \zeta_\alpha \rangle : \alpha < \lambda \Big\rangle$$

is constructed. At stage $\alpha < \lambda$ of the game:

⋆ Generic chooses a non-zero ordinal $\zeta_\alpha < \mu_\alpha$ and then

⋆ the two players play a subgame of length ζ_α alternately choosing successive terms of a sequence $\langle p^\alpha_\xi, q^\alpha_\xi : \xi < \zeta_\alpha \rangle$. At a stage $\xi < \zeta_\alpha$ of the subgame, first Generic picks a condition $p^\alpha_\xi \in \mathbb{Q}$ and then Antigeneric answers with a condition q^α_ξ stronger than p^α_ξ.

At the end, Generic wins the play $\Big\langle \zeta_\alpha, \langle p^\alpha_\xi, q^\alpha_\xi : \xi < \zeta_\alpha \rangle : \alpha < \lambda \Big\rangle$ of $\Game^{\rm rc\mathbf{a}}_{\bar{\mu}}(p, \mathbb{Q})$ if and only if

$(\circledast)^{\rm rc}_{\mathbf{a}}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} (\forall \alpha < \lambda)(\exists \xi < \zeta_\alpha)(q^\alpha_\xi \in \mathunderscore{G}_{\mathbb{Q}}).$$

Definition 4

We say that a forcing notion $\mathbb{Q}$ is *reasonably A–bounding over $\bar{\mu}$* if

(a) $\mathbb{Q}$ is strategically $({<}\lambda)$–complete, and

(b) for any $p \in \mathbb{Q}$, Generic has a winning strategy in the game $\Game^{\rm rcA}_{\bar{\mu}}(p, \mathbb{Q})$.

In an analogous manner we define when the forcing notion $\mathbb{Q}$ is *reasonably* **a***–bounding over $\bar{\mu}$*.

If $\mu_\alpha = \lambda$ for each $\alpha < \lambda$, then we may omit $\bar{\mu}$ and say *reasonably A–bounding* etc.

Theorem 5 (Cf [RoSh 860, Thm 3.2], [RoSh 890, Thm 3.13])

Assume that $\lambda, \bar{\mu}$ *are as declared before and* $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ *is a* λ*–support iteration such that for every* $\xi < \gamma$*,*

$$\Vdash_{\mathbb{P}_\xi} \text{“}\,\mathbb{Q}_\xi \textit{ is reasonably A–bounding over } \bar{\mu}\,\text{”}.$$

Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ *is reasonably* **a***–bounding over* $\bar{\mu}$ *(and actually more).*

Observation 6

If $\mathbb{Q}$ *is is reasonably* **a***–bounding, then it is* λ*–proper and* ${}^\lambda\lambda$*–bounding.*

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Remark 7

- The forcing notion $\mathbb{Q}^2_{\varphi,\bar{F}}$ is reasonably A–bounding. (Note: since λ is strongly inaccessible the forcing notions $\mathbb{Q}^2_{\varphi,\bar{F}}$ and $\mathbb{Q}^3_{\varphi,\bar{F}}$ are equivalent.)
- The forcing $\mathbb{P}^*$ (Goldstern–Shelah type) is reasonably **a**–bounding but it is very **not** A–bounding! The iterations as in Theorem 5 preserve some sort of ultrafilters on λ while $\mathbb{P}^*$ destroys them, see [RoSh 890].
- We have also nicely double **a**–bounding forcing and this property is preserved in λ–support iterations. It is “almost” weaker then being reasonably **a**–bounding (well, we need to add a demand that the conditions played by Generic in the subgames are pairwise incompatible).

Better stuff: the Bs

The A-like bounding properties do not cover forcing notions of the type $\mathbb{Q}^{\ell,\bar{E}}$ or $\mathbb{Q}^{1,\bar{E}}_{E}$ (as those add unbounded λ–reals). We will cover $\mathbb{Q}^{\ell,\bar{E}}$ in the third part, at the moment let us look at bounding properties weak enough to cover $\mathbb{Q}^{1,\bar{E}}_{E}$.
We will weaken the winning conditions in the game $\Game^{\rm rcA}_{\bar{\mu}}(p,\mathbb{Q})$ — instead of

$(\circledast)^{\rm rc}_{\rm A}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \big(\forall \alpha < \lambda\big)\big(\exists t \in I_\alpha\big)\big(q^\alpha_t \in \underset{\sim}{G}_{\mathbb{Q}}\big)$$

we will demand something like

$(\circledast)^{\rm rc}_{\rm X}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \big\{\alpha < \lambda : \big(\exists t \in I_\alpha\big)\big(q^\alpha_t \in \underset{\sim}{G}_{\mathbb{Q}}\big)\big\} \in \text{Some Filter.}$$

What are the choices for the filter?

University of Nebraska Omaha

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We will weaken the winning conditions in the game $\Game^{\mathrm{rcA}}_{\bar{\mu}}(p,\mathbb{Q})$ — instead of

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What are the choices for the filter?

First Choice

Observation 8

If a forcing notion $\mathbb{Q}$ is strategically $(<\lambda)$–complete and D is a normal filter on λ, then the normal filter generated by D in $\mathbf{V}^{\mathbb{Q}}$ is proper.

Abusing notation, we may denote the normal filter generated by D in $\mathbf{V}^{\mathbb{Q}}$ also by D or by $D^{\mathbb{Q}}$. Thus if $\underset{\sim}{A}$ is a $\mathbb{Q}$–name for a subset of λ, then $p \Vdash_{\mathbb{Q}} \underset{\sim}{A} \in D^{\mathbb{Q}}$ if and only if for some $\mathbb{Q}$–names $\underset{\sim}{A}_\alpha$ for elements of $D^{\mathbf{V}}$ we have that $p \Vdash_{\mathbb{Q}} \underset{\alpha<\lambda}{\triangle} \underset{\sim}{A}_\alpha \subseteq \underset{\sim}{A}$ (where $\triangle$ denotes the operation of diagonal intersection).

From now on, in addition to previous assumptions, we suppose

(g) D is a normal filter on λ.

(Intention: D is "orthogonal" to E.)

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If a forcing notion $\mathbb{Q}$ is strategically $({<}\lambda)$–complete and D is a normal filter on λ, then the normal filter generated by D in $\mathbf{V}^{\mathbb{Q}}$ is proper.

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University of Nebraska Omaha

Games B & b

Games $\Game^{\rm rcB}_{D,\bar{\mu}}(p,\mathbb{Q}), \Game^{\rm rc\mathbf{b}}_{D,\bar{\mu}}(p,\mathbb{Q})$ are defined similarly to $\Game^{\rm rcA}_{\bar{\mu}}(p,\mathbb{Q}), \Game^{\rm rc\mathbf{a}}_{\bar{\mu}}(p,\mathbb{Q})$, except that the winning criterions are now

$(\circledast)^{\rm rc}_{\rm B}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \text{“}\ \big\{\alpha<\lambda : \big(\exists t \in I_\alpha\big)\big(q^\alpha_t \in \underset{\sim}{G}_{\mathbb{Q}}\big)\big\} \in D^{\mathbb{Q}}\ \text{”},$$

and

$(\circledast)^{\rm rc}_{\mathbf{b}}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \text{“}\ \big\{\alpha<\lambda : \big(\exists \xi < \zeta_\alpha\big)\big(q^\alpha_\xi \in \underset{\sim}{G}_{\mathbb{Q}}\big)\big\} \in D^{\mathbb{Q}}\ \text{”},$$

respectively.

Definition 9

A strategically $(<\lambda)$–complete forcing notion $\mathbb{Q}$ is *reasonably B–bounding (–bounding, respectively) over* $D, \bar{\mu}$ if for any $p \in \mathbb{Q}$, Generic has a winning strategy in the game $\Game^{\rm rcB}_{D,\bar{\mu}}(p, \mathbb{Q})$ ($\Game^{\rm rc\mathbf{b}}_{D,\bar{\mu}}(p, \mathbb{Q})$, respectively).

Theorem 10 ([RoSh 860, Thm 3.1])

Assume that $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathop{\mathbb{Q}}\limits_{\sim}{}_\xi : \xi < \gamma \rangle$ *is a* λ*–support iteration such that for every* $\xi < \gamma$,

$$\Vdash_{\mathbb{P}_\xi} \text{“}\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi \text{ is reasonably B–bounding over } D, \bar{\mu}\text{”}.$$

Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ *is reasonably* $\mathbf{b}$*–bounding over* $D, \bar{\mu}$ *(and so also* λ*–proper).*

Unfortunately, while there are examples of forcings which are B– but not $\mathbf{a}$–bounding, this does not help us with $\mathbb{Q}^{1,\bar{E}}_{E}$.

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Better Choice

Definition 11

A $\mathcal{D}\ell$–parameter on λ is a triple $\mathbf{p} = (\bar{P}, S, D) = (\bar{P}^{\mathbf{p}}, S^{\mathbf{p}}, D^{\mathbf{p}})$ such that

- D is a proper uniform normal filter on λ, $S \in D$,
- $\bar{P} = \langle P_\delta : \delta \in S \rangle$ and $P_\delta \in [{}^{\delta}\delta]^{<\lambda}$ for each $\delta \in S$,
- for every function $f \in {}^{\lambda}\lambda$ we have

$$\mathrm{set}^{\mathbf{p}}(f) \stackrel{\mathrm{def}}{=} \{\delta \in S : f{\restriction}\delta \in P_\delta\} \in D.$$

Remember that our λ is strongly inaccessible. Let D is the filter generated by club subsets of λ and $P_\delta = {}^{\delta}\delta$, $\bar{P} = \langle P_\delta : \delta < \lambda \rangle$. Then $(\bar{P}, \lambda, D)$ is a $\mathcal{D}\ell$–parameter on λ.

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Definition 12

Let **p** be a $\mathcal{D}\ell$–parameter on λ and $\mathbb{Q}$ be a a strategically $({<}\lambda)$–complete forcing notion. In $\mathbf{V}^{\mathbb{Q}}$ we define

- $D^{\mathbf{p}}[\mathbb{Q}] = D^{\mathbf{p}[\mathbb{Q}]}$ is the normal filter generated by $D^{\mathbf{p}} \cup \{\mathrm{set}^{\mathbf{p}}(f) : f \in {}^{\lambda}\lambda\}$, and
- $\mathbf{p}[\mathbb{Q}] = (\bar{P}^{\mathbf{p}}, S^{\mathbf{p}}, D^{\mathbf{p}[\mathbb{Q}]})$.

The filter $D^{\mathbf{p}}[\mathbb{Q}]$ is (potentially) larger than $D^{\mathbb{Q}}$ (the normal filter generated by D), but it is still a proper filter:

Lemma 13

Assume that $\mathbf{p} = (\bar{P}, S, D)$ *is a* $\mathcal{D}\ell$*–parameter on* λ *and* $\mathbb{Q}$ *is a strategically* $({<}\lambda)$*–complete forcing notion. Then* $\Vdash_{\mathbb{Q}} \emptyset \notin D^{\mathbf{p}}[\mathbb{Q}]$. *Consequently,* $\Vdash_{\mathbb{Q}}$ "$\mathbf{p}[\mathbb{Q}]$ *is a* $\mathcal{D}\ell$*–parameter on* λ".

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Let $\mathbf{p} = (\bar{P}, S, D)$ be a $\mathcal{D}\ell$–parameter on λ and $\mathbb{Q}$ be a forcing notion. For a condition $p \in \mathbb{Q}$ we define a game $\partial^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ between two players, Generic and Antigeneric, as follows. A play of $\partial^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ lasts λ steps and during a play a sequence $\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha\rangle : \alpha < \lambda \Big\rangle$ is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now,

$(\aleph)_\alpha$ first Generic chooses a set I_α of cardinality $< \lambda$ and a system $\langle p^\alpha_t : t \in I_\alpha\rangle$ of conditions from $\mathbb{Q}$,

$(\beth)_\alpha$ then Antigeneric answers by picking a system $\langle q^\alpha_t : t \in I_\alpha\rangle$ of conditions from $\mathbb{Q}$ such that $(\forall t \in I_\alpha)(p^\alpha_t \leq q^\alpha_t)$.

At the end, Generic wins the play $\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha\rangle : \alpha < \lambda\rangle$ of $\partial^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ if and only if

$(\circledast)^{\mathbf{p}}_{\rm rbB}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \text{“}\,\big\{\alpha < \lambda : \big(\exists t \in I_\alpha\big)\big(q^\alpha_t \in \Gamma_{\mathbb{Q}}\big)\big\} \in D[\mathbb{Q}]\,\text{”}.$$

Let $\mathbf{p} = (\bar{P}, S, D)$ be a $\mathcal{D}\ell$–parameter on λ and $\mathbb{Q}$ be a forcing notion. For a condition $p \in \mathbb{Q}$ we define a game $\beth^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ between two players, Generic and Antigeneric, as follows. A play of $\beth^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ lasts λ steps and during a play a sequence $\left\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle : \alpha < \lambda \right\rangle$ is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now,

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Let $\mathbf{p} = (\bar{P}, S, D)$ be a $\mathcal{D}\ell$–parameter on λ and $\mathbb{Q}$ be a forcing notion. For a condition $p \in \mathbb{Q}$ we define a game $\supset^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ between two players, Generic and Antigeneric, as follows. A play of $\supset^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ lasts λ steps and during a play a sequence $\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha\rangle : \alpha < \lambda \Big\rangle$ is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now,

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At the end, Generic wins the play $\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha\rangle : \alpha < \lambda \Big\rangle$ of $\supset^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ if and only if

$(\circledast)^{\mathbf{p}}_{\rm rbB}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \text{“ } \big\{\alpha < \lambda : \big(\exists t \in I_\alpha\big)\big(q^\alpha_t \in \Gamma_{\mathbb{Q}}\big)\big\} \in D[\mathbb{Q}] \text{ ”}.$$

Let $\mathbf{p} = (\bar{P}, S, D)$ be a $\mathcal{D}\ell$–parameter on λ and $\mathbb{Q}$ be a forcing notion. For a condition $p \in \mathbb{Q}$ we define a game $\eth^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ between two players, Generic and Antigeneric, as follows. A play of $\eth^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ lasts λ steps and during a play a sequence $\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle : \alpha < \lambda \Big\rangle$ is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now,

$(\aleph)_\alpha$ first Generic chooses a set I_α of cardinality $< \lambda$ and a system $\langle p^\alpha_t : t \in I_\alpha \rangle$ of conditions from $\mathbb{Q}$,

$(\beth)_\alpha$ then Antigeneric answers by picking a system $\langle q^\alpha_t : t \in I_\alpha \rangle$ of conditions from $\mathbb{Q}$ such that $(\forall t \in I_\alpha)(p^\alpha_t \leq q^\alpha_t)$.

At the end, Generic wins the play $\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle : \alpha < \lambda \rangle$ of $\eth^{\rm rbB}_{\mathbf{p}}(p, \mathbb{Q})$ if and only if

$(\circledast)^{\mathbf{p}}_{\rm rbB}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \text{“ } \big\{\alpha < \lambda : \big(\exists t \in I_\alpha\big)\big(q^\alpha_t \in \Gamma_{\mathbb{Q}}\big)\big\} \in D[\mathbb{Q}] \text{ ”}.$$

Definition 14

Let $\mathbb{Q}$ be a strategically $({<}\lambda)$–complete forcing notion. We say that $\mathbb{Q}$ is *reasonably B–bounding over* $\mathbf{p}$ if for any $p \in \mathbb{Q}$, Generic has a winning strategy in the game $\Game^{\mathrm{rbB}}_{\mathbf{p}}(p, \mathbb{Q})$.

Theorem 15 ([RoSh 888, Thm 1.10])

Assume that λ is a strongly inaccessible cardinal and $\mathbf{p}$ is a $\mathcal{D}\ell$–parameter on λ. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ be a λ–support iteration such that for every $\alpha < \lambda$,
$\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is reasonably B–bounding over $\mathbf{p}[\mathbb{P}_\alpha]$ ".
Then

(a) $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ *is λ–proper,*

(b) *if $\underset{\sim}{\tau}$ is a $\mathbb{P}_\gamma$–name for a function from λ to $\mathbf{V}$, $p \in \mathbb{P}_\gamma$, then there are $q \geq p$ and $\langle A_\xi : \xi < \lambda \rangle$ such that $(\forall \xi < \lambda)(|A_\xi| < \lambda)$ and*

$$q \Vdash \text{"} \{\xi < \lambda : \underset{\sim}{\tau}(\xi) \in A_\xi\} \in D^{\mathbf{p}}[\mathbb{P}_\gamma] \text{ ".}$$

Proposition 16 ([RoSh 888, Prop. 1.12])

Let $\mathbf{p} = (\bar{P}, S, D)$ *be a* $\mathcal{D}\ell$*–parameter on* λ *such that* $\lambda \setminus S \in E$.

1. $\mathbb{Q}^{1,\bar{E}}_E$ *is reasonably B–bounding over* $\mathbf{p}$.
2. *If (*λ *is strongly inaccessible and)* $(\forall \delta \in S)(|P_\delta| \leq |\delta|)$*, then* $\Vdash_{\mathbb{Q}^{1,\bar{E}}_E} D^{\mathbb{Q}^{1,\bar{E}}_E} \neq D[\mathbb{Q}^{1,\bar{E}}_E]$.

Many different $\mathbb{Q}_E^{1,E}$'s — oh my lords (of iteration)!

Definition 17

1. *A forcing notion with λ–complete (κ, μ)–purity* is a triple $(\mathbb{Q}, \leq, \leq_{\rm pr})$ such that $\leq, \leq_{\rm pr}$ are transitive reflexive (binary) relations on $\mathbb{Q}$ such that
   - (a) $\leq_{\rm pr} \subseteq \leq$,
   - (b) both $(\mathbb{Q}, \leq)$ and $(\mathbb{Q}, \leq_{\rm pr})$ are strategically $(<\lambda)$–complete,
   - (c) for every $p \in \mathbb{Q}$ and a $(\mathbb{Q}, \leq)$–name $\underset{\sim}{\tau}$ for an ordinal below κ, there are a set A of size less than μ and a condition $q \in \mathbb{Q}$ such that $p \leq_{\rm pr} q$ and q forces (in $(\mathbb{Q}, \leq)$) that "$\underset{\sim}{\tau} \in A$".
2. If $(\mathbb{Q}, \leq, \leq_{\rm pr})$ is a forcing notion with λ–complete (κ, μ)–purity for every κ, then we say that it has *λ–complete $(*, \mu)$–purity*.

Let $\mathbb{Q} = (\mathbb{Q}, \leq, \leq_{\rm pr})$ be a forcing notion with λ–complete $(*, \lambda^+)$–purity, $\mathbf{p} = (\bar{P}, S, D)$ be a $\mathcal{D}\ell$–parameter on λ, $\mathcal{U}$ be a normal filter on λ and $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ be a sequence of cardinals below λ.

Many different $\mathbb{Q}_E^{1,E}$'s — oh my lords (of iteration)!

Definition 17

1. *A forcing notion with λ–complete (κ, μ)–purity* is a triple $(\mathbb{Q}, \leq, \leq_{\rm pr})$ such that $\leq, \leq_{\rm pr}$ are transitive reflexive (binary) relations on $\mathbb{Q}$ such that
   (a) $\leq_{\rm pr} \subseteq \leq$,
   (b) both $(\mathbb{Q}, \leq)$ and $(\mathbb{Q}, \leq_{\rm pr})$ are strategically $({<}\lambda)$–complete,
   (c) for every $p \in \mathbb{Q}$ and a $(\mathbb{Q}, \leq)$–name $\underaccent{\tilde}{\tau}$ for an ordinal below κ, there are a set A of size less than μ and a condition $q \in \mathbb{Q}$ such that $p \leq_{\rm pr} q$ and q forces (in $(\mathbb{Q}, \leq)$) that "$\underaccent{\tilde}{\tau} \in A$".
2. If $(\mathbb{Q}, \leq, \leq_{\rm pr})$ is a forcing notion with λ–complete (κ, μ)–purity for every κ, then we say that it has *λ–complete $(*, \mu)$–purity*.

Let $\mathbb{Q} = (\mathbb{Q}, \leq, \leq_{\rm pr})$ be a forcing notion with λ–complete $(*, \lambda^+)$–purity, $\mathbf{p} = (\bar{P}, S, D)$ be a $\mathcal{D}\ell$–parameter on λ, $\mathcal{U}$ be a normal filter on λ and $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ be a sequence of cardinals below λ.

A game with purity

For $p \in \mathbb{Q}$ we define a game $\Game^{\rm pr}_{\mathcal{U},\mathbf{p},\bar{\mu}}(p,\mathbb{Q})$ between Generic and Antigeneric. A play of $\Game^{\rm pr}_{\mathcal{U},\mathbf{p},\bar{\mu}}(p,\mathbb{Q})$ lasts λ steps and during the play a sequence $\Big\langle \ell_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in \mu_\alpha\rangle : \alpha < \lambda \Big\rangle$ is constructed.

At a stage $\alpha < \lambda$ of the game:

$(\aleph)^{\rm pr}_\alpha$ first Antigeneric pics $\ell_\alpha \in \{0,1\}$.

$(\beth)^{\rm pr}_\alpha$ After this, Generic chooses a system $\langle p^\alpha_t : t \in \mu_\alpha\rangle$ of pairwise incompatible conditions from $\mathbb{Q}$, and

$(\gimel)^{\rm pr}_\alpha$ Antigeneric answers with a system of conditions $q^\alpha_t \in \mathbb{Q}$ (for $t \in \mu_\alpha$) such that for each $t \in \mu_\alpha$:

- $p^\alpha_t \le q^\alpha_t$, and
- if $\ell_\alpha = 1$, then $p^\alpha_t \le_{\rm pr} q^\alpha_t$.

At the end, Generic wins the play

$$\Big\langle \ell_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in \mu_\alpha\rangle : \alpha < \lambda \Big\rangle$$

if and only if either $\{\alpha < \lambda : \ell_\alpha = 1\} \notin \mathcal{U}$, or

$(\circledast)^{\mathbf{p}}_{\rm pr}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that

$$p^* \Vdash_{\mathbb{Q}} \text{“ } \Big\{\alpha < \lambda : \big(\exists t \in \mu_\alpha\big)\big(q^\alpha_t \in \Gamma_{\mathbb{Q}}\big)\Big\} \in D[\mathbb{Q}] \text{ ”}.$$

Definition 18

We say that the forcing notion $\mathbb{Q}$ (with λ–complete $(*, \lambda^+)$–purity) is *purely B^*–bounding over $\mathcal{U}, \mathbf{p}, \bar{\mu}$* if for any $p \in \mathbb{Q}$, Generic has a winning strategy in the game $\partial^{\rm pr}_{\mathcal{U},\mathbf{p},\bar{\mu}}(p, \mathbb{Q})$.

Observation 19

1. *For $p, q \in \mathbb{Q}^{1,\bar{E}}_E$ let $p \leq_{\rm pr} q$ mean that $p \leq q$ and $\mathrm{root}(p) = \mathrm{root}(q)$. Then $(\mathbb{Q}^{1,\bar{E}}_E, \leq, \leq_{\rm pr})$ is a forcing notion with λ–complete $(*, \lambda^+)$–purity.*
2. *Assume that $\mathbf{p} = (\bar{P}, S, D)$ is a $\mathcal{D}\ell$–parameter on λ and $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ is a sequence of non-zero cardinals below λ such that $(\forall \alpha \in S)(|P_\alpha| \leq \mu_\alpha)$. Then $(\mathbb{Q}^{1,\bar{E}}_E, \leq, \leq_{\rm pr})$ is purely B^*–bounding over $E, \mathbf{p}, \bar{\mu}$.*

Note **no** demands on E being orthogonal to D!

Definition 18

We say that the forcing notion $\mathbb{Q}$ (with λ–complete $(*, \lambda^+)$–purity) is *purely B^*–bounding over $\mathcal{U}, \mathbf{p}, \bar{\mu}$* if for any $p \in \mathbb{Q}$, Generic has a winning strategy in the game $\Game^{\rm pr}_{\mathcal{U},\mathbf{p},\bar{\mu}}(p, \mathbb{Q})$.

Observation 19

1. *For $p, q \in \mathbb{Q}^{1,\bar{E}}_E$ let $p \leq_{\rm pr} q$ mean that $p \leq q$ and $\mathrm{root}(p) = \mathrm{root}(q)$. Then $(\mathbb{Q}^{1,\bar{E}}_E, \leq, \leq_{\rm pr})$ is a forcing notion with λ–complete $(*, \lambda^+)$–purity.*
2. *Assume that $\mathbf{p} = (\bar{P}, S, D)$ is a $\mathcal{D}\ell$–parameter on λ and $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ is a sequence of non-zero cardinals below λ such that $(\forall \alpha \in S)(|P_\alpha| \leq \mu_\alpha)$. Then $(\mathbb{Q}^{1,\bar{E}}_E, \leq, \leq_{\rm pr})$ is purely B^*–bounding over $E, \mathbf{p}, \bar{\mu}$.*

Note **no** demands on E being orthogonal to D!

Theorem 20 ([RoSh 888, Thm 2.7])

Assume that

1. λ *is strongly inaccessible,* $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ *is a sequence of cardinals below* λ*,* $\mathbf{p} = (\bar{P}, S, D)$ *is a* $\mathcal{D}\ell$*–parameter on* λ*, and*
2. $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ *is a* λ*–support iteration,*
3. $\underset{\sim}{\mathcal{U}}_\alpha$ *is a* $\mathbb{P}_\alpha$*–name for a normal filter on* λ *(for* $\alpha < \gamma$*),*
4. $A_{\alpha,\beta} \subseteq \lambda$ *is such that* $\Vdash_{\mathbb{P}_\alpha} A_{\alpha,\beta} \in \underset{\sim}{\mathcal{U}}_\alpha$ *and* $\Vdash_{\mathbb{P}_\beta} \lambda \setminus A_{\alpha,\beta} \in \underset{\sim}{\mathcal{U}}_\beta$ *(for* $\alpha < \beta < \gamma$*), and*
5. *for every* $\alpha < \gamma$*,*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\underset{\sim}{\mathbb{Q}}_\alpha \text{ is purely } B^*\text{–bounding over } \underset{\sim}{\mathcal{U}}_\alpha, \mathbf{p}[\mathbb{P}_\alpha], \bar{\mu}\text{”.}$$

Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ *is* λ*–proper.*

Thank You!

Thank you for your attention today.
I hope you will come to the third part of this series. We will talk about two ways to "cover" forcing $\mathbb{Q}^{2,\bar{E}}$ *and others — with and without diamonds.*

References

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